ROUND I Number theory

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Explore powers of 3. Determine the units digit of 3^{22}.

2. $11.11_2 + 101.101_2 = ?_{10}$

3. The greatest common divisor of a pair of numbers is 315 and the least common multiple of the pair is $3^2 \cdot 5^3 \cdot 7^2$. Find both pairs of whole numbers which satisfy these conditions

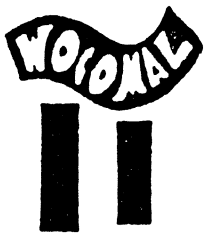
ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____10_____

(3 pts) 3 _____and_____

Assabet Valley, St. John's, Shrewsbury



ROUND II: Algebra 1 - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Evaluate this expression if $a = -9; g = \frac{2}{3}; i = 8; o = -6; p = \frac{1}{2}; r = -0.8; s = \sqrt{121}; t = -12$:

   go-patriots

2. In the equation $2x + 3y = 24$, the variables x and y are non-negative integers. Find the number of ordered pairs (x,y) which satsify this equation.

3. A bank teller was to change a $20 bill into quarters and dimes. She made an error and interchanged the number of quarters and the number of dimes, thereby paying out $9 more than she should have. How many dimes should she have paid out?

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

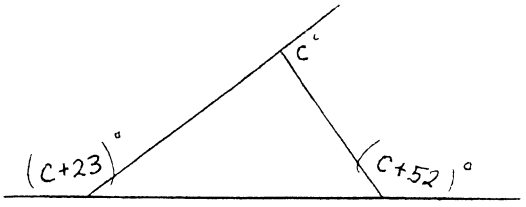
(3 pts) 3. _____

Bromfield, Tahanto, Westborough

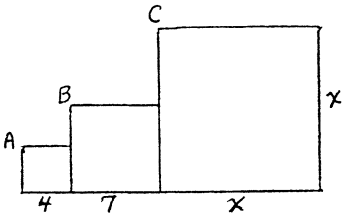
ROUND III: Geometry - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the value of c:



2. In the squares shown, the vertices A, B, and C lie in a striaght line. Find the value of length x.



3. The vertices of an equilateral triangle are the midpoints of the sides of a larger equilateral triangle. A circle is inscribed in the smaller triangle. Find the exact ratio of the area of the circle to the area of the larger triangle. Do not approximate π or any radicals involved.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Hudson, Shrewsbury, Tahanto

ROUND IV: Logs, exponents, radicals

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Express in simplest form: $3 \cdot 8^{\frac{2}{3}} \cdot \left(\frac{27}{8}\right)^{\frac{-1}{3}}$

2. Solve the radical equation $\sqrt{x+6} + x = 14$

3. Solve for x: $\log_2 x \cdot \log_4 x \cdot \log_6 x = \log_2 x \cdot \log_4 x + \log_2 x \cdot \log_6 x + \log_4 x \cdot \log_6 x$

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Bromfield, Tahanto, Westborough

March 31, 1999 Wocomal Varsity Meet

ROUND V: Trigonometry - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM, EXCEPT NUMBER 1

1. An airplane leaves the runway climbing at 18^o on a straight path with a constant speed of 285 feet per second. Find the altitude of the plane after one minute, to the nearest foot.

2. A sector of a circle has area 25 sq cm and central angle 0.5 radian. Find the radius and arc length Include units!

3. Solve for x: $\sin[(\text{arc cos } x) - 30^o] = x$. (arc cos x is the same as $\cos^{-1} x$ or inverse cos x)

ANSWERS

(1 pt) 1. _____ ft _____

(2 pts) 2. radius _____ arc length _____

(3 pts) 3. _____

Holy Name, Shrewsbury, Westborough

TEAM ROUND Topics of previous rounds and open

ALL ANWERS MUST BE IN SIMPLEST EXACT FORM UNLESS SPECIFIED OTHERWISE IN THE PROBLEM

2 POINTS EACH

1 The three digit number 2A3 is added to the number 326 to give the three digit number 5B9 If 5B9 is divisible by 9, then A+B must equal what?

2 At the present time, the sum of the ages of a father and his son is 33 years Find the smallest whole number of years until the father's age is 4 times the son's age

3 A circle, containing point A and tangent to the x-axis, rolls without slipping along the x-axis, starting with point A at the origin If after 3.5 revolutions point A is at (π, y), find y

4 If $x > y > 0$, list by letter those conclusions which are always true "a" may be any real number

A $x + a > y + a$

B $\log_{\frac{1}{2}} x > \log_{\frac{1}{2}} y$

C $x(\log_{10} 0.1) > y(\log_{10} 0.1)$

D $1/x > 1/y$

E $a^x > a^y$

F $\sqrt{x} > \sqrt{y}$

5 For $0 \le x \le \pi$, at how many points do the graphs of $y = \sin(60x)$ and $y = \tan(30x)$ intersect?

6 Write the next term in this geometric sequence $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[6]{2}$,

7 If the arithmetic mean of x and y is 6 and their geometric mean is 5, write a quadratic equation in $x^2 + bx + c = 0$ form that has x and y as its roots

8 How long, to the nearest day, is 10^1 seconds?

9 How many points (x,y) on or inside the circle $x^2 + y^2 = 50$ have nothing but integer coordinates?

March 31, 1999 Wocomal Varsity Meet Answers

Round I # thry

1. 1 pt 9
2. 2 pts 9.375 or $9\frac{3}{8}$
3. 3 pts 315, 55125 and 2205, 7875 order does not matter

Round II alg 1

1. 1 pt 273, 711.2
2. 2 pts 5
3. 3 pts 100

Round III geom

1. 1 pt 95 or 95°
2. 2 pts 12.25 or $12\frac{1}{4}$ or $\frac{49}{4}$
3. 3 pts $\frac{\pi}{12\sqrt{3}}$ or $\frac{\pi\sqrt{3}}{36}$ or in a:b form

Round IV logs exp rad

1. 1 pt 8
2. 2 pts 10
3. 3 pts 1 & 48 need both and or comma OK

Round V trig

1. 1 pt 5284
2. 2 pt r = 10 cm arc = 5 cm need both and units
3. 3 pts $\frac{1}{2}$ or .5

Team Round 2 pts each

1. 6
2. 1
3. $\frac{2}{7}$
4. A, F
5. 91
6. 1
7. $x^2 - 12x + 25 = 0$ needed
8. 42
9. 161

March 31, 1999 WOCOMAL VARSITY MEET BRIEF SOLUTIONS

ROUND I

1. $3^0 = 1$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$

exponent: 0, 1, 2, 3, 4, 5, 6, 7, ... 22

units digit: 1, 3, 9, 7, 1, 3, 9, 7, ... (9)

2. $2 + 1 + \frac{1}{2} + \frac{1}{4} = 3\frac{3}{4}$
$4 + 1 + \frac{1}{2} + \frac{1}{8} = 5\frac{5}{8}$
sum $= 9\frac{3}{8}$

3. GCD $= 315 = 3^2 5\ 7$
LCM $= 3^2 \cdot 5^3\ 7^2$
Pairs are of the form $3^2 \cdot 5 \cdot 7\ x$ and $3^2 \cdot 5 \cdot 7\ y$ where x and y have no common factor > 1 and $x^2 y \le 5^2 \cdot 7$
Two possibilities: $x = 5^2 \cdot 7$, $y = 1$
$x = 5^2$, $y = 7$
One pair
$3^2 \cdot 5 \cdot 7 \cdot 5^2 \cdot 7 = 55{,}125$ and $3^2 \cdot 5 \cdot 7 \cdot 1 = 315$
Other pair
$3^2 \cdot 5 \cdot 7 \cdot 5^2 = 7{,}875$ and $3^2 \cdot 5 \cdot 7\ 7 = 2{,}205$

ROUND II

1. $\left(\frac{2}{3}\right)(-6) - \left(\frac{1}{2}\right)(-9)(-12)(-0.8)(8)(-6)(-12)(11)$
$= -4 + 273{,}715.2 = 273{,}711.2$

2.

possible Y	0	1	2	3	4	5	6	7	8
resulting x	12	N	9	N	6	N	3	N	0

N = not integer. 5 pairs

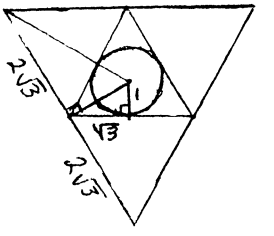
3. Right: $25Q + 10D = 2000$
Error: $10Q + 25D = 2900$
$50Q + 125D = 14500$
$50Q + 20D = 4000$
$105D = 10500$
$D = 100$

ROUND III

1. Ext angle sum $= 360°$
$3c + 75 = 360 \Rightarrow c = 95$

2. Slope of $\overline{AB}$ = slope of $\overline{BC}$
$\frac{3}{4} = \frac{x-7}{7} \Rightarrow x = 12\frac{1}{4}$
OR use similar triangles

3. Let circle radius $= 1$, use 30, 60, 90 $\triangle$s



$\frac{\pi 1^2}{\frac{(4\sqrt{3})^2\sqrt{3}}{4}} = \frac{\pi}{12\sqrt{3}}$

ROUND IV

1. $3 \cdot \sqrt[3]{8}^2 \cdot \sqrt[3]{\frac{8}{27}} = 3 \cdot 4 \cdot \frac{2}{3} = 8$

2. $\sqrt{x+6} = 14 - x$
$x + 6 = 196 - 28x + x^2$
$0 = x^2 - 29x + 190$
$= (x-10)(x-19)$
$x = 10$ ✓ since $4 = 14 - 10$
$x = \cancel{19}$ since $5 \neq 14 - 19$

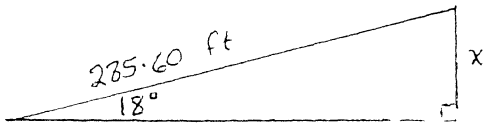
3. Express all logs in terms of the same base, ln is convenient

$$\frac{\ln x}{\ln 2}\cdot\frac{\ln x}{\ln 4}\cdot\frac{\ln x}{\ln 6} = \frac{\ln x}{\ln 2}\cdot\frac{\ln x}{\ln 4} + \frac{\ln x}{\ln 2}\cdot\frac{\ln x}{\ln 6} + \frac{\ln x}{\ln 4}\cdot\frac{\ln x}{\ln 6}$$

Mult by $\ln 2 \cdot \ln 4 \ln 6$ to get
$(\ln x)^3 = (\ln x)^2(\ln 6 + \ln 4 + \ln 2)$
If $\ln x = 0$, $x = 1$
Otherwise
$\ln x = \ln(6 \cdot 4 \cdot 2)$ and $x = 48$

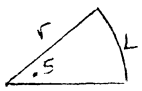
ROUND V

1.



$\sin 18° = \frac{x}{285\ 60} \Rightarrow x \approx 5284$ ft

2



Area $\frac{25}{\pi r^2} = \frac{0.5}{2\pi}$

$50 = .5r^2 \Rightarrow r = 10$ cm

Arc length $\frac{L}{2\pi 10} = \frac{0.5}{2\pi} \Rightarrow L = 5$ cm

3 $\sin(A-B) = \sin A \cos B - \cos A \sin B$,

$\sin(\arccos x) = \sqrt{1-x^2}$, $\cos(\arccos x) = x$

$\therefore \sqrt{1-x^2}\ \frac{\sqrt{3}}{2} - x\ \frac{1}{2} = x$

$\sqrt{1-x^2} \cdot \sqrt{3} \qquad 3x$

$1 - x^2 = 3x^2 \Rightarrow x = \frac{1}{2}$ or ~~$\frac{1}{2}$~~ ← doesn't check

4 A yes add a

B no , $x=2, y=1$ gets $-1>0$

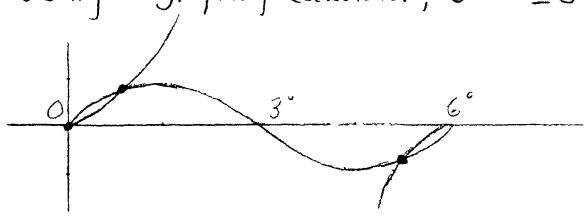
C no , becomes $-x > -y$

D no $x=2, y=1$ gets $\frac{1}{2} > 1$

L no $a=-1, x=3$ $y=2$ gets $-1>1$

F yes , since positive

5 Using a graphing calculator, $0° \le \theta \le 6°$



3 intersection pts for $0° < \theta < 6°$

$3 \cdot \frac{180}{6} = 90$ pts for $0° \le \theta < 180°$

and 91 for $0 < x \le \pi$

6 $2^{\frac{1}{2}}, 2^{\frac{1}{3}}, 2^{\frac{1}{6}}, \ldots$

$2^{\frac{1}{2}} r = 2^{\frac{1}{3}} \Rightarrow r = \frac{2^{\frac{1}{3}}}{2^{\frac{1}{2}}} = 2^{-\frac{1}{6}}$

Then $2^{\frac{1}{6}} 2^{-\frac{1}{6}} = 2^0 = 1$

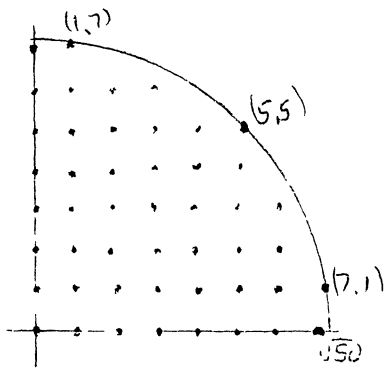
7 $x+11$, , , — $= 5$

$x^2 - (\text{root sum})x + (\text{root product}) = 0$

$x^2 - 12x + 25 = 0$

8 $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3\ 2 \cdot 1 \text{ sec}}{\frac{60 \text{ sec}}{\text{min}} \cdot \frac{60 \text{ min}}{\text{hr}} \cdot \frac{24 \text{ hr}}{\text{day}}} = 42$ days

9



33 in each quadrant for 132 plus 29 on the axes, making 161

TEAM ROUND

1. $B \le 7$ and $14 - B$ is

589 ∴ $B = 4$, $A = 2$ and $A + B = 6$

2 Let F = father's age, Y = years sought

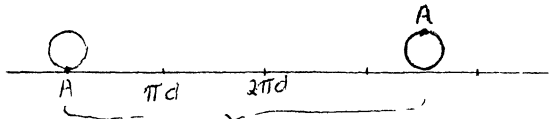
$F + Y = 4(33 - F + Y)$

$5F - 132 = 3Y > 0$

So $5F > 132$ and $F > 26.4$

If $F = 27$, $2Z = 4.7$ and we get smallest $Y = 1$

3



~~π~~ = (3.5)(circumference)

y = diam of circle

$3.5\pi y = \pi \Rightarrow y = \frac{2}{7}$